3. Transformações Canônicas

PGF 5005 - Mecânica Clássica
web.if.usp.br/controle

(Referências principais: Lichtenberg e Lieberman, 1992, Percival, 1989, Lowenstein 2012)

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Transformações Canônicas

(Mudanças de variáveis com equações de Hamilton para as novas variáveis)

$$\boldsymbol{q}, \boldsymbol{p} \rightarrow \bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}$$

$$H(\boldsymbol{p}, \boldsymbol{q}, t) \rightarrow \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)$$

$$\bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t) = H(\boldsymbol{p}, \boldsymbol{q}, t) + \frac{\partial}{\partial t} F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$$

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad \bar{q}_i = \frac{\partial F_2}{\partial \bar{p}_i}$$

2N equações entre as novas e as antigas variáveis

$$\dot{p}_i = -\frac{\partial H}{\partial q_i},$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$

Analogamente, podemos introduzir geratrizes

$$F_3(\boldsymbol{p}, \bar{\boldsymbol{q}}, t), \qquad F_4(\boldsymbol{p}, \bar{\boldsymbol{p}}, t)$$

$$F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t)$$

Obter as transformações

Princípio variacional

$$\delta \int L\,dt = 0 \quad \rightarrow \quad \delta\left[\int_{t_1}^{t_2}\left(\sum_i p_i\dot{q}_i - H(\boldsymbol{p}, \boldsymbol{q}, t)\right)dt\right] = 0$$

$L = L' + dF_1/dt$

$$\sum_i p_i\dot{q}_i - H(\boldsymbol{p}, \boldsymbol{q}, t) = \sum_i \bar{p}_i\dot{\bar{q}}_i - \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t) + \frac{d}{dt}F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t)$$

A derivada temporal da **função geratriz** F_1 é:

$$\frac{d}{dt}F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t) = \sum_i \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_1}{\partial \bar{q}_i}\dot{\bar{q}}_i + \frac{\partial F_1}{\partial t}$$

Comparando as duas equações anteriores obtemos $p_i = \dfrac{\partial F_1}{\partial q_i}$ $\bar{p}_i = -\dfrac{\partial F_1}{\partial \bar{q}_i}$

Além das equações, $p_i = \frac{\partial F_1}{\partial q_i} \qquad \bar{p}_i = -\frac{\partial F_1}{\partial \bar{q}_i}$

obtemos também

$$\bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t) = H(\boldsymbol{p}, \boldsymbol{q}, t) + \frac{\partial}{\partial t} F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t)$$

Introduzindo a função, $F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t) = F_1(\boldsymbol{q}, \bar{\boldsymbol{q}}, t) + \sum_i \bar{q}_i \bar{p}_i$

obtemos

$$p_i = \frac{\partial F_2}{\partial q_i} \quad \bar{q}_i = \frac{\partial F_2}{\partial \bar{p}_i}$$

$$\bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t) = H(\boldsymbol{p}, \boldsymbol{q}, t) + \frac{\partial}{\partial t} F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t)$$

Aplicação: Uma possibilidade para obter F_2

(em H separável)

$\boldsymbol{q}, \boldsymbol{p} \rightarrow \bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}$ $H(\boldsymbol{p}, \boldsymbol{q}, t) \rightarrow \bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t)$

$$\bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}, t) = H(\boldsymbol{p}, \boldsymbol{q}, t) + \frac{\partial}{\partial t} F_2(\boldsymbol{q}, \bar{\boldsymbol{p}}, t) = 0$$

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad \bar{q}_i = \frac{\partial F_2}{\partial \bar{p}_i}$$

$$H\left(\frac{\partial F_2}{\partial \boldsymbol{q}}, \boldsymbol{q}, t\right) + \frac{\partial F_2}{\partial t} = 0$$

Se essa equação obtida for integrável obtem-se F_2 e as variáveis em função do tempo.

Outra aplicação: Uma possibilidade para obter F_2

H autonoma (independe de t)

$\boldsymbol{q}, \boldsymbol{p}$ ➔ $\bar{\boldsymbol{q}}, \bar{\boldsymbol{p}}$

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad \bar{q}_i = \frac{\partial F_2}{\partial \bar{p}_i}$$

$\bar{H}(\bar{\boldsymbol{p}}, \bar{\boldsymbol{q}}) = H(\boldsymbol{p}, \boldsymbol{q}) = E$

➔ $$H\left(\frac{\partial F_2}{\partial \boldsymbol{q}}, \boldsymbol{q}\right) = E$$

Equação de Hamilton-Jacobi.

Se essa equação obtida for separável, ela será integrada e F_2 obtida. Nesse caso, as variáveis podem ser obtidas em função do tempo.

Duas Lagrangianas que diferem por um derivada df/dt descrevem o mesmo movimento

$$\bar{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t)$$

$$= L + \dot{q}\frac{\partial f}{\partial q} + \frac{\partial f}{\partial t},$$

Escrevendo $$\frac{\partial \bar{L}}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{q}} + \frac{\partial f}{\partial q}, \quad \frac{\partial \bar{L}}{\partial q} = \frac{\partial L}{\partial q} + \frac{d}{dt}\left(\frac{\partial f}{\partial q}\right)$$

Obtemos $$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}}\right) - \frac{\partial \bar{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

Podemos escrever a mesma equação de Lagrange

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}}\right) - \frac{\partial \bar{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

De $\quad p = \partial L/\partial \dot{q} \qquad \bar{p} = \partial \bar{L}/\partial \dot{q}$

obtemos $\quad \bar{p} = p + \dfrac{\partial f}{\partial q}$

Equação de Newton

Lagrangiana de uma partícula $L(q, \dot{q}, t) = \frac{1}{2} m\dot{q}^2 - V(q, t)$.

$$\frac{\partial L}{\partial \dot{q}} = m\dot{q}, \quad \frac{\partial L}{\partial q} = - \frac{\partial V}{\partial q}$$

Equação de Lagrange $$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}$$

Equação de Newton $$m\ddot{q} = - \frac{\partial V}{\partial q}$$

Hamiltoniana

$$H(\boldsymbol{p}, \boldsymbol{q}, t) \equiv \sum_i \dot{q}_i p_i - L(\dot{\boldsymbol{q}}, \boldsymbol{q}, t)$$

$$L(q, \dot{q}, t) = \tfrac{1}{2} m\dot{q}^2 - V(q, t)$$

$$p = \partial L / \partial \dot{q}$$

$$H(q, p, t) = \frac{p^2}{2m} + V(q, t)$$

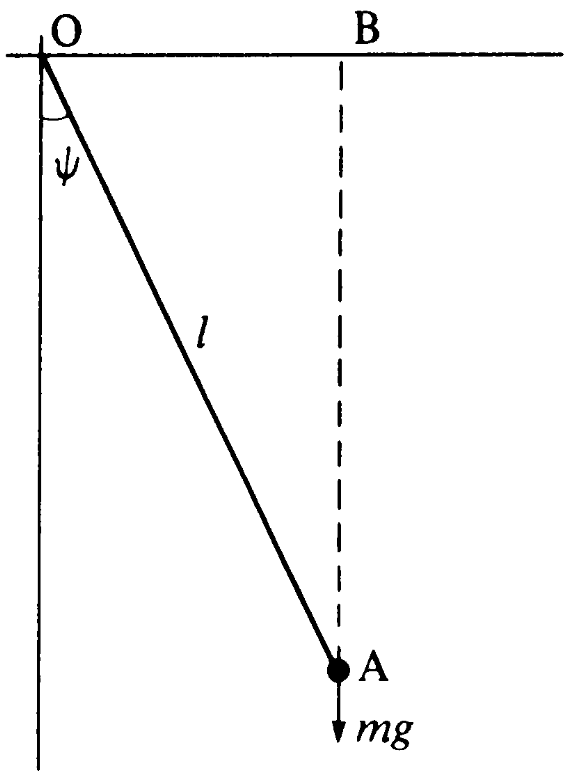
L – T – V   **H = T + V**

Percival, Introduction to Dynamics

Example 5.5

Find the Lagrangian and Hamiltonian for a pendulum moving in a vertical plane.

Here the configuration of the system may be defined by the angle between the pendulum and the downward vertical, ψ. The generalized velocity $u = \dot{\psi}$ is the angular velocity of OA.



The kinetic energy of the mass m at A is

$$T = \tfrac{1}{2}m\,(\text{speed})^2 = \tfrac{1}{2}mw^2 = \tfrac{1}{2}ml^2u^2$$

and the potential energy is

$$V = -\,mg\,\text{AB} = -\,mgl\cos\psi,$$

so that the Lagrangian is

$$L(\psi, u) = \tfrac{1}{2}ml^2u^2 + mgl\cos\psi.$$

The momentum conjugate to ψ is

$$p = \frac{\partial L}{\partial u} = ml^2u \quad \text{or} \quad u = \dot{\psi} = \frac{1}{ml^2}\,p.$$

The Hamiltonian, from Equation (5.16), is

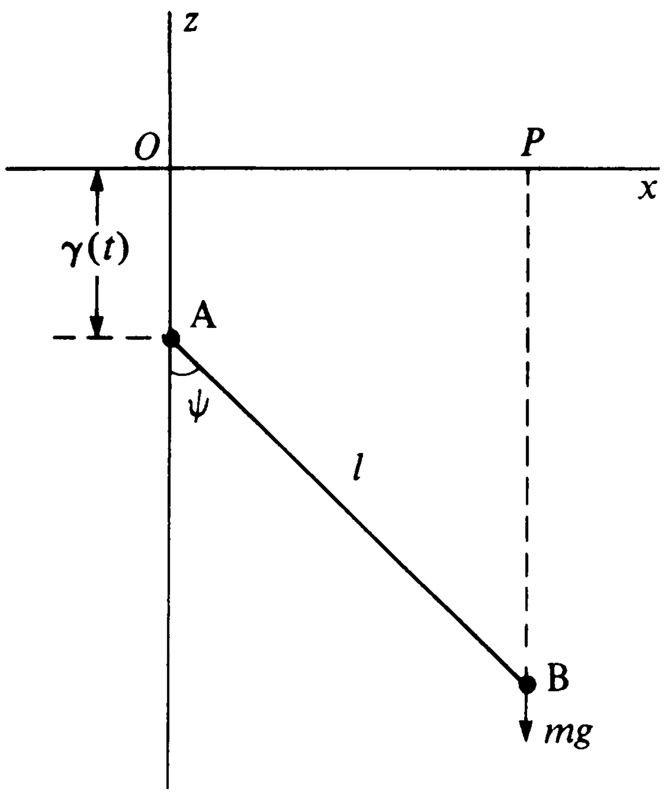
$$H(\psi, p) = pu - \left(\tfrac{1}{2}ml^2u^2 + mgl\cos\psi\right) \qquad \left(u = \frac{p}{ml^2} \right),$$

$$= \frac{p^2}{2ml^2} - mgl\cos\psi.$$

Example 5.7

Find the Lagrangian and Hamiltonian of a pendulum comprising a mass m attached to a light, stiff rod AB of length l free to move in a vertical plane, and such that the end A of the rod is forced to move vertically, its distance from a fixed point O being a given function $\gamma(t)$ of time.

The generalized coordinate is the angle ψ between AB and the downward vertical. In order to determine the speed of the mass, it is helpful to write dowr its coordinates with respect to the axes Ox and Oz shown in the diagram.



These are

$$x = l \sin \psi,$$

$$z = - l \cos \psi - \gamma(t) .$$

The potential energy is

$$V(z, t) = mgz = - mg \, [l \cos \psi + \gamma(t)]$$

and the kinetic energy is

$$T = \tfrac{1}{2} m \, (\text{speed})^2 = \tfrac{1}{2} m \, (\dot{x} + \dot{z})^2$$

$$= \tfrac{1}{2} m \, [l^2 \dot{\psi}^2 \cos^2 \psi + (l \dot{\psi} \sin \psi - \dot{\gamma})^2]$$

$$= \tfrac{1}{2} m \, [l^2 \dot{\psi}^2 - 2l \dot{\psi} \dot{\gamma} \sin \psi + \dot{\gamma}^2] .$$

Thus the Lagrangian is

$$L = \tfrac{1}{2} m \, (l^2 \dot{\psi}^2 - 2l \dot{\psi} \dot{\gamma} \sin \psi) + mgl \cos \psi + h(t),$$

where

$$h(t) = \tfrac{1}{2} m \dot{\gamma}^2 + mg \gamma$$

The momentum conjugate to ψ is

$$p = \frac{\partial L}{\partial \dot{\psi}} = m\,(l^2 \dot{\psi} - l\,\dot{\gamma} \sin \psi)$$

and, after some manipulation, we find that the Hamiltonian is

$$H(\psi, p, t) = \frac{(p + ml\,\dot{\gamma} \sin \psi)^2}{2ml^2} - mgl \cos \psi .$$

An alternative, but more convenient, Hamiltonian may be obtained by using the result of example 5.4. On writing

$$\dot{\psi}\,\dot{\gamma} \sin \psi = -\frac{\mathrm{d}}{\mathrm{d}t}\,(\dot{\gamma} \cos \psi) + \ddot{\gamma} \cos \psi$$

we obtain

$$\bar{L} = \tfrac{1}{2} m l^2 \dot{\psi}^2 + m l (g - \ddot{\gamma}) \cos\psi,$$

showing that the vertical acceleration has the same effect as a time-varying gravitational field. The conjugate momentum is now

$$\bar{p} = \frac{\partial \bar{L}}{\partial \dot{\psi}} = m l^2 \dot{\psi}$$

and the Hamiltonian,

$$\bar{H} = \frac{\bar{p}^2}{2 m l^2} - m l (g - \ddot{\gamma}) \cos\psi.$$

Funções Geradoras de Transformações Canônicas

Relações entre as antigas e as novas variáveis

type 1: $$p_i = \frac{\partial F_1}{\partial q_i}(q, Q, t), \qquad P_i = -\frac{\partial F_1}{\partial Q_i}(q, Q, t),$$

type 2: $$p_i = \frac{\partial F_2}{\partial q_i}(q, P, t), \qquad Q_i = \frac{\partial F_2}{\partial P_i}(q, P, t),$$

type 3: $$q_i = -\frac{\partial F_3}{\partial p_i}(p, Q, t), \qquad P_i = -\frac{\partial F_3}{\partial Q_i}(p, Q, t),$$

type 4: $$q_i = -\frac{\partial F_4}{\partial p_i}(p, P, t), \qquad Q_i = \frac{\partial F_4}{\partial P_i}(p, P, t).$$

Relações entre Funções Geradoras

$$F_1(q, Q, t) = F_2(q, P, t) - P \cdot Q,$$

$$F_3(p, Q, t) = F_1(q, Q, t) - p \cdot q,$$

$$F_4(p, P, t) = F_2(q, P, t) - p \cdot q.$$

Sendo F_2 conhecida, obtemos F_1: $F_1(q, Q, t) = F_2(q, P, t) - P \cdot Q$

Derivando com relação a q_k
com Q,, t fixos

$$\frac{\partial F_1}{\partial q_k} = \frac{\partial F_2}{\partial q_k} + \frac{\partial F_2}{\partial P_j}\frac{\partial P_j}{\partial q_k} - \frac{\partial P_j}{\partial q_k}Q_j$$

Como sabemos que

$$p_i = \frac{\partial F_2}{\partial q_i}(q, P, t), \qquad Q_i = \frac{\partial F_2}{\partial P_i}(q, P, t)$$

Obtemos a) $\frac{\partial F_1}{\partial q_k} = p_k$

Analogamente $\frac{\partial F_1}{\partial Q_k} = \frac{\partial F_2}{\partial P_j}\frac{\partial P_j}{\partial Q_k} - \frac{\partial P_j}{\partial Q_k}Q_j - P_k$

Obtemos b) $\frac{\partial F_1}{\partial Q_k} = -P_k$

Equações a, b relacionam q, p com Q, P

De

$$F_1(q, Q, t) = F_2(q, P, t) - P \cdot Q,$$
$$F_3(p, Q, t) = F_1(q, Q, t) - p \cdot q,$$
$$F_4(p, P, t) = F_2(q, P, t) - p \cdot q.$$

obtemos

$$\frac{\partial F_1}{\partial t}(q, Q, t) = \frac{\partial F_2}{\partial t}(q, P, t) = \frac{\partial F_3}{\partial t}(p, Q, t) = \frac{\partial F_4}{\partial t}(p, P, t)$$

Exemplos de Transformações

$$F_2(q, P) = q_k P_k, \qquad p_i = \frac{\partial F_2}{\partial q_i} = P_i, \qquad Q_i = \frac{\partial F_2}{\partial P_i} = q_i,$$

$$F_1(q, Q) = q_k Q_k, \qquad p_i = \frac{\partial F_1}{\partial q_i} = Q_i, \qquad P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i,$$

$$F_2(q, P) = f_k(q) P_k, \qquad p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial f_k(q)}{\partial q_i} P_k, \qquad Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q).$$